Separability in terms of a single entanglement witness

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Separability problem is formulated in terms of a characterization of a *single* entanglement witness operator. More specifically, we show that any (in general multipartite) state ϱ is separable if and only if a specially constructed entanglement witness W_ϱ (which may always be chosen decomposable) is weakly optimal, i.e., its expectation value vanishes on at least one product vector. This changes the conceptual aspect of the separability problem and rises some new questions about properties of positive maps.

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Some of the fundamental problems in quantum information theory concern detection and characterization of entanglement. In many instances, questions concerning detection can be successfully addressed via theory of positive maps. There, separability (i.e., absence of entanglement) of $\varrho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is equivalent to the statement that for all positive maps Λ acting on $B(\mathcal{H}_A)$, operator $\sigma = [\Lambda \otimes I_B](\varrho)$ is positive [1, 2]. Via Jamiołkowski's isomorphism [3], the last statement can be reformulated in terms of physical (Hermitian) operators instead of positive maps as follows [2]:

State ϱ is separable when for every Hermitian operator $W \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\langle\alpha|\langle\beta|W|\alpha\rangle|\beta\rangle \geq 0$ for all products $|\alpha\rangle|\beta\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the following non-negativity condition is satisfied

$$\langle W\rangle_\varrho := \mathrm{Tr}(W\varrho) \geq 0 \tag{1}$$

Importance of this formulation was first recognized by Terhal [4, 5]. She coined the term 'entanglement witness' for these operators W, which have at least one negative eigenvalue (a state, for which some $\langle W\rangle_\varrho$ is negative, is then clearly entangled). Subsequently she utilized the possibility of experimental entanglement tests via verification of condition (1) in a laboratory. Since then entanglement witnesses became one of the most popular tools for entanglement detection, as they allow to identify entanglement without otherwise difficult to avoid complete state tomography [6, 7, 8] (for nonlinear entanglement detection methods see for instance [9]). Many facts about the set of entanglement witnesses are then known today [10] and impressive experimental implementations have been performed [11].

Despite all the progress, practical characterization of the set of witnesses, which would provide precise optimization parameters is still eluding the researches. Usually, the parameters can only be estimated with limited accuracy [5, 12] and the witnesses have a structure, which is not easy to handle.

As part of the effort to improve on this unsatisfactory situation, in this paper we simplify the conceptual aspect of the separability problem at a cost of the size of the underlying Hilbert space. We consider a given decomposition of a $d_A \otimes d_B$ state ϱ on $\mathcal{H}_A \otimes \mathcal{H}_B$ and construct an associated entanglement witness W_ϱ on $\mathcal{H}' \otimes \mathcal{H}'$ with $\dim \mathcal{H}' = N \leq ((d_A d_B)^2 - 1)^2$. Weak optimality of this witness (for the optimality notion of entanglement witness see [12, 13]) is then proven to be equivalent to separability of the original state ϱ (we call a witness weakly optimal if its expectation value vanishes on at least one product vector).

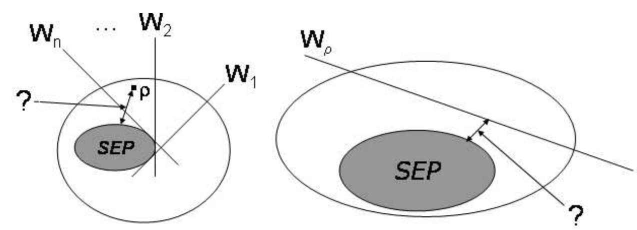

FIG. 1: The separability problem originally expressed in terms of infinitely many entanglement witnesses (left) is here proved to be equivalent to weak optimality of a single entanglement witness in a larger Hilbert space (right).

Our approach has the following conceptual advantage: since the witness W_ϱ can be explicitly calculated, all the elements of the possible subsequent tests have well-defined and clear structures. In particular, *arbitrary multipartite* separability problem is here mapped into analysis of *a single bipartite* entanglement witness (see Fig. 1). Moreover, our formulation provokes some new interesting questions about the structure of the set of the entanglement witnesses and the corresponding maps derived from a given quantum state.

In this context, it is worth noticing that the question of strict positivity of a single entanglement witness on separable states has an algorithmic solution in terms of the so called Henkel forms. The underlying algorithm was constructed more than three decades ago by Jamiołkowski [14]. Even though Jamiołkowski's algorithm is not of practical use here, it is still conceptually interesting. In particular, up to our knowledge, this is the first algorithm, which can decide witness' optimality in a finite,

a'priori known number of steps.

Our state witness W_ϱ is constructed from the biconcurrence matrix [15], two forms reflecting its transformation properties and an additional projection. We begin the construction with a decomposition of ϱ in terms of subnormalized vectors, so that $\varrho = \sum_i |\Psi_i\rangle\langle\Psi_i|$ (eigendecomposition is usually the most obvious although by no means necessary choice). The decomposition defines the corresponding biconcurrence matrix $B = B(\varrho)$ [15]. It can be most easily expressed as [16]:

$$B_{m\mu,n\nu} = \langle\Psi^m_{AB}|\langle\Psi^\mu_{A'B'}|P^{asym}_{AA'} \otimes P^{asym}_{BB'}|\Psi^n_{AB}\rangle|\Psi^\nu_{A'B'}\rangle \tag{2}$$

with P^{asym} being a projector onto the antisymmetric subspace.

When one begins with the eigendecomposition of ϱ, then matrix B is an operator acting in a $d\otimes d$-dimensional space ($d = d_A d_B$ is the number of eigenvectors of ϱ). In order to allow for a separable decomposition whenever it exists, one has to extend this space to $N\otimes N$ dimensions, where $N = d^2 - 1$ (If a separable decomposition of ϱ exists, then it may require up to N elements). So, from now on we will regard B as a matrix on the extended space $\mathcal{H}\otimes\mathcal{H}$ with $\dim\mathcal{H} = N$.

Matrix B is positive. Moreover, it is symmetric with respect to the transposition of indices m and μ, as well as n and ν. It is related to separability of ϱ via the following theorem (see [15]):

Theorem 1 *State ϱ is separable if and only if the following function (we call it biconcurrence function):*

$$\mathcal{B}(\varrho) = \inf_U \sum_m [U\otimes UBU^\dagger \otimes U^\dagger]_{mm,mm} \tag{3}$$

vanishes. The infimum in the definition of $\mathcal{B}(\varrho)$ is taken over all unitary matrices U acting on $\mathcal{H}$.

Each unitary matrix represents an orthonormal basis. We can therefore rewrite definition (3) as

$$\mathcal{B}(\varrho) = \inf_{\{|x_i\rangle\}} \sum_{i=1}^N \langle x_i|\langle x_i|B|x_i\rangle|x_i\rangle = 0, \tag{4}$$

where the infimum is taken over all orthonormal bases $\{|x_i\rangle\}$. Consequently, we can rewrite the quoted theorem as follows.

A bipartite state ϱ is separable if and only if there is a set of vectors $|x_i\rangle$, $i = 1\ldots N$, for which the following three forms vanish at the same time
(i) zero form condition:

$$G_0 = \sum_{i=1}^N \langle x_i|\langle x_i|B|x_i\rangle|x_i\rangle = 0, \tag{5}$$

(ii) orthogonality condition for $|x_i\rangle$ which gives:

$$G_1 = \sum_{ij}[\langle x_i|x_j\rangle\langle x_j|x_i\rangle - \delta_{ij}||x_i||^2||x_j||^2] = 0, \tag{6}$$

(iii) normalization condition for the basis $|x_i\rangle$

$$G_2 = \sum_i [(\sum_j ||x_j||^2)/N - ||x_i||^2]^2 = 0. \tag{7}$$

Since all $G_0, G_1, G_2 \geq 0$, the three conditions can be replaced by a single $\alpha G_0 + \beta G_1 + \gamma G_2 = 0$, for any fixed $\alpha, \beta, \gamma > 0$. In other words, a state is entangled if and only if

$$\alpha G_0 + \beta G_1 + \gamma G_2 > 0 \tag{8}$$

for all sets of vectors $\{|x_i\rangle\}$.

To convert this into a property of a witness operator, we need to extend the Hilbert space once more. Operator B is defined on $\mathcal{H}\otimes\mathcal{H}$. We extend each $\mathcal{H}$ to $\mathcal{H}\otimes\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is an auxiliary space isomorphic to $\mathcal{H}$. One can then notice that every vector $|u\rangle \in \mathcal{H}\otimes\tilde{\mathcal{H}}$ can be written in the form

$$|u\rangle = \sum_i |x_i\rangle|i\rangle. \tag{9}$$

This observation allows one to substitute single vectors in the extended space for the sets of vectors in the conditions (5-7). To this end, we define two auxiliary operators: $P_{cl} = \sum_i |i\rangle|i\rangle\langle i|\langle i|$ (classically correlated projector) and $V = \sum_{ij} |i\rangle|j\rangle\langle j|\langle i|$ (swap operator). These operators will act on $\tilde{\mathcal{H}}\otimes\tilde{\mathcal{H}}$. For clarity, we mark this action by a tilde on top of the relevant operator.

With this notation, we can rewrite the necessary and sufficient condition for entanglement (8) in terms of a new degree four form A as

$$\mathcal{B}(\varrho) = \min_u \langle u|\langle u|A|u\rangle|u\rangle > 0. \tag{10}$$

The minimum is now taken over all vectors $|u\rangle \in \mathcal{H}\otimes\tilde{\mathcal{H}}$. Operator A acts on $(\mathcal{H}\otimes\tilde{\mathcal{H}})\otimes(\mathcal{H}\otimes\tilde{\mathcal{H}})$ and is given by:

$$\begin{aligned} A &= \alpha B\otimes\tilde{P}_{cl} + \beta(I\otimes\tilde{V} - I\otimes\tilde{P}_{cl}) \\ &+\gamma(I\otimes\tilde{P}_{cl} - \frac{1}{N} I\otimes\tilde{I}) = \alpha A_0 + \beta A_1 + \gamma A_2. \end{aligned} \tag{11}$$

Parameters $\alpha, \beta, \gamma > 0$ here can be chosen at will. This freedom may be utilized for, e.g., optimization of the numerical separability tests based on condition (10).

Each of the three terms contributing to operator A has non-negative expectation values on product vectors $|uu\rangle$. Neither the whole operator nor any of its parts is, however, a witness. A_0 is non-negative. Its addition to a witness constructed out of A_1 and A_2 makes the witness weaker. When the outcome represents a witness which is so weak, that it is not even weakly optimal, then the corresponding state ϱ is entangled. Operators A_1 and A_2 do not represent entanglement witnesses since they

have negative expectation values on some product vectors $|uv\rangle$ with $|u\rangle \neq |v\rangle$. One can remove this disadvantage without effecting the expectation values on products $|uu\rangle$ by adding to A a projection on the antisymmetric space $P^{asym} = (1/2)(I \otimes \tilde{I} - V \otimes \tilde{V})$ with large enough weight. Moreover, without effecting the expectation values $\langle u|\langle u|A|u\rangle|u\rangle$ in (10), one may substitute $Y = P^{sym} A P^{sym}$ for the original operator A. When one has done the latter, then the following lemma gives a straightforward method to calculate a weight with which P^{asym} has to be added to an operator like A_1 or A_2 to guarantee its conversion into an entanglement witness.

Lemma .- *Let X be a Hermitian operator on a product Hilbert space $\mathcal{H} \otimes \mathcal{H}$ such that $X = P^{sym} X P^{sym}$ and $\forall |u\rangle \in \mathcal{H}$, $\langle u|\langle u|X|u\rangle|u\rangle \geq 0$. Moreover let $X_C = X + CP^{asym}$, where P^{asym} projects onto the antisymmetric subspace of $\mathcal{H} \otimes \mathcal{H}$. Then: (i) $C \geq ||X||_\infty \Rightarrow \langle u|\langle v|X_C|u\rangle|v\rangle \geq 0$ (ii) $C \geq 2||X||_\infty \Rightarrow \forall |u\rangle, |v\rangle\ \exists g$ such that:*

$$\begin{aligned} \langle u|\langle v|X_C|u\rangle|v\rangle \ &\geq\ \langle g|\langle g|X_C|g\rangle|g\rangle \\ &\geq\ \mathcal{X} := \inf_u \langle u|\langle u|X|u\rangle|u\rangle. \end{aligned} \tag{12}$$

The latter property implies in particular that $\mathcal{X}_C := \inf_{u,v} \langle u|\langle v|X_C|u\rangle|v\rangle = \mathcal{X}$. In other words, if we select a sufficiently large C, then the expectation value of X_C on a separable state is always an *upper bound* for the infimum in the definition of $\mathcal{X}$.

Proof.- Take any two normalized vectors $|u\rangle$, $|v\rangle$. Symmetry $X = P^{sym} X P^{sym}$ implies that

$$\langle u|\langle v|X|u\rangle|v\rangle = \langle \Psi|X|\Psi\rangle, \tag{13}$$

where $|\Psi\rangle := (|u\rangle|v\rangle + |v\rangle|u\rangle)/2$. Apart from an unimportant global phase factor, vector $|v\rangle$ can be decomposed into $|v\rangle = a|u\rangle + b|u^\perp\rangle$ with $a, b \geq 0$ and $a^2 + b^2 = 1$. Consequently, $|\Psi\rangle = a|u\rangle|u\rangle + b(|u\rangle|u^\perp\rangle + |u^\perp\rangle|u\rangle)/2$ and $||\Psi||^2 = a^2 + b^2/2$. Finally, one can easily convince oneself that the Schmidt decomposition of vector $|\Psi\rangle$ reads $|\Psi\rangle = x|e\rangle|e\rangle + y|f\rangle|f\rangle$, with $x = (1+a)/2$, $y = (1-a)/2$. All this allows us to write:

$$\begin{aligned} &\langle u|\langle v|X|u\rangle|v\rangle = \langle \Psi|X|\Psi\rangle \\ &= x^2 \langle e|\langle e|X|e\rangle|e\rangle + y^2 \langle f|\langle f|X|f\rangle|f\rangle \\ &+2xyRe(\langle e|\langle e|A|f\rangle|f\rangle) \\ &\geq x^2 \langle e|\langle e|X|e\rangle|e\rangle + y^2 \langle f|\langle f|X|f\rangle|f\rangle \\ &-2xy|\langle e|\langle e|X|f\rangle|f\rangle| \\ &\geq x^2 \langle e|\langle e|X|e\rangle|e\rangle + y^2 \langle f|\langle f|X|f\rangle|f\rangle - 2xyC \\ &\geq -(1-a^2)C/2 = -(1 - |\langle u|v\rangle|^2)C/2 \\ &= -C\langle u|\langle v|P^{asym}|u\rangle|v\rangle \end{aligned} \tag{14}$$

for any $C \geq ||X||_\infty$.

Comparison of the first and the last expression in (14) immediately gives:

$$\langle u|\langle v|X + CP^{asym}|u\rangle|v\rangle \geq 0, \quad C \geq ||X||_\infty \tag{15}$$

which proves property (i).

For (ii), we return to the third last line in (14). It does not exceed $(x^2+y^2)\langle \tilde{e}|\langle \tilde{e}|X|\tilde{e}\rangle|\tilde{e}\rangle - 2xyC$, where $|\tilde{e}\rangle$ denotes this of the two vectors ($|e\rangle$ or $|f\rangle$), which produces the lower expectation value for X. Consequently,

$$\langle u|\langle v|X|u\rangle|v\rangle \geq (x^2+y^2)\langle \tilde{e}|\langle \tilde{e}|X|\tilde{e}\rangle|\tilde{e}\rangle - 2xyC \tag{16}$$

which can be rewritten as:

$$\langle u|\langle v|X|u\rangle|v\rangle + 4xyC \geq (x^2+y^2)\langle \tilde{e}|\langle \tilde{e}|X|\tilde{e}\rangle|\tilde{e}\rangle + 2xyC. \tag{17}$$

Utilizing the fact that $C \geq \langle \tilde{e}|\langle \tilde{e}|X|\tilde{e}\rangle|\tilde{e}\rangle$ on the RHS of (17), we arrive at:

$$\langle u|\langle v|X|u\rangle|v\rangle + 4xyC \geq (x+y)^2 \langle \tilde{e}|\langle \tilde{e}|X|\tilde{e}\rangle|\tilde{e}\rangle \tag{18}$$

Finally we use the relations: $x + y = 1$ and $4xy = 2\langle u|\langle v|P^{asym}|u\rangle|v\rangle$ and obtain:

$$\langle u|\langle v|X + 2CP^{asym}|u\rangle|v\rangle \geq \langle \tilde{e}|\langle \tilde{e}|X|\tilde{e}\rangle|\tilde{e}\rangle. \tag{19}$$

After changing form $2C \geq ||X||_\infty$ to $C \geq 2||X||_\infty$ this gives (12) which concludes the proof.

Our matrix Y satisfies the assumptions of the lemma. Consequently $Y_C = Y + CP^{asym}$ ($C \geq 2||Y||_\infty$) is a good candidate for a witness operator. In fact, it is a witness, since it has at least one negative eigenvalue. In this way we have arrived at our central result:

Theorem 2 .- *A bipartite state ϱ is separable if and only if its corresponding entanglement witness $W_\varrho = Y_C$ with $C > ||Y||_\infty$ is weakly optimal. Moreover if $C \geq 2||Y||_\infty$ then the witness satisfies in addition the condition (12), guaranteeing that $\langle u|\langle v|W_\varrho|u\rangle|v\rangle \geq \mathcal{B}(\varrho)$ for all $|u\rangle$, $|v\rangle$.*

We need a sharp inequality in the first condition for C above to secure that $\langle u|\langle v|W_\varrho|u\rangle|v\rangle > 0$ for all $|u\rangle \neq |v\rangle$.

A simple corollary to this theorem provides a direct link between separable states and weakly optimal entanglement witnesses in $\mathcal{C}^{n^2} \otimes \mathcal{C}^{n^2}$, namely:

Corollary 1 .- *Every separable state with a pure state product decomposition of rank n generates a corresponding weakly optimal entanglement witness in $\mathcal{C}^{n^2} \otimes \mathcal{C}^{n^2}$.*

Clearly, the strongest witnesses constructed in this way are those for $A_0 = 0$. Even then, however, the witness construction based on the lemma, although universal, does not have to produce the most interesting witnesses in their own rights. To illustrate this point, we consider the choice $\beta = \gamma = 1$ and put $A_0 = 0$. The resulting operator A is then $A_{12} = I \otimes \tilde{V} - (1/N) I \otimes \tilde{I}$. Its symmetrization is $Y = (1/2)[I \otimes \tilde{V} + V \otimes \tilde{I} - (1/N)(I \otimes \tilde{I} + V \otimes \tilde{V})]$. With a little bit of work, one can easily check that $\|Y\|_\infty = (N+1)/N$. According to the lemma, one then needs to add $[(N+1)/N]P^{asym}$ to Y, in order to secure its conversion into an entanglement witness W^{sym}. Apparently, this is quite unnecessary. Knowing that for any operator $X \in B(\mathcal{C}^n)$, $\|X\|_{\mathrm{Tr}} \leq \sqrt{n}\,\|X\|_{\mathrm{HS}}$ (symbols $\|\cdot\|_{\mathrm{Tr}}$ and $\|\cdot\|_{\mathrm{HS}}$ stand for trace and Hilbert-Schmidt

norm respectively), one can easily show that without any symmetrization, it is enough to add $(2/N)P^{asym}$ to A_{12} in order to convert it into a witness operator $W = I \otimes \tilde{V} - (1/N)V \otimes \tilde{V}$. It is easy to see that W belongs to the class of so called decomposable witnesses (see [12]). Witnesses like W^{sym} and W may still have zero expectation values on some product vectors $|uv\rangle$ with $|u\rangle \neq |v\rangle$. For that, they do not make any good ground for entanglement identification in ϱ. To remedy this disadvantage, it is, however, enough to add P^{asym} with any positive weight to such a witness (see the comment after Theorem 2). This will not change the witness' expectation value on products $|uu\rangle$. On the other hand, the new witness (let us denote it by W_+^{sym} and W_+) will become strictly positive on all products $|uv\rangle$ with $|u\rangle \neq |v\rangle$. This is enough to guarantee that after the addition of the contribution from A_0, the resulting witness will be weakly optimal if and only if the state ϱ, from which A_0 (via B) is derived, is separable [17].

Our method of linking separability of a bipartite state to weak optimality of a single entanglement witness readily generalizes for the states shared by many parties. In the latter case, however, different aspects of separability are described by different matrices B [18]. Thus, one will end up with different corresponding operators A_0, depending on which aspect of multi-partite entanglement/separability one would like to test. Nevertheless, the design and structure of the state-independent contributions to our witness (A_1 and A_2) as well as condition (10), together with (11), will be exactly as in the bipartite case, irrespectively of the number of parties sharing the tested state ϱ. Consequently, the design and the properties of W_ϱ for multi-partite ϱ will be exactly the same as in the bipartite case.

Connection to the theory of positive maps .- Via Jamiołkowski isomorphism, the relation between bipartite states and their 'state witnesses' directly translates into a relation between bipartite states and positive but not completely positive maps. In particular, it is easy to see that in the isomorphism, operators, which are not weakly optimal, are mapped onto fully mixing maps. These are the maps which transform any state into a positive matrix of full rank. We then have another immediate corollary to Theorem 2.

Corollary 2 .- *A bipartite state ϱ is entangled if and only if a positive map Λ_ϱ (it can be chosen to be decomposable) is fully mixing.*

Indeed, our choice of parameters ($\beta = \gamma$) produces clearly decomposable witnesses and, consequently, decomposable maps.

Separability problem is known to be computationally hard [19]. Nevertheless, analysis of the properties of witnesses W_ϱ (and maps Λ_ϱ) should be at least in some cases relatively straightforward. One can then hope that our approach not only sheds new light on the conceptual aspect of the separability problem, but also may become a starting point for development of new, more efficient numerical separability tests. Finally, allowing for $\beta \neq \gamma$ in formula (11) may lead to nondecomposable witnesses and nondecomposable maps. This in turn may lead to some new questions about the nature of these witnesses, their possible relation to potential bound entanglement in ϱ or their ability to reveal new geometrical properties of the boundary of the set of separable states. We leave these questions for further research.

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